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Energy landscape of neural networks storing spatially correlated patterns

E A Dorotheyev†, G Rotundo‡ and B Tirozzi‡

† Landau Institute for Theoretical Physics, 117940, Moscow, Kosygina 2, Russia
‡ Department of Mathematics, University of Rome 'La Sapienza', Piazza Aldo Moro 2, 00185 Rome, Italy

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Abstract. Hopfield-like neural networks with spatially organized data are studied by a mean-field theory. The internal structure of the data is described by a matrix $\hat{C}$ whose elements C_{ij} are equal to the correlation between two pixels, i and j, of any input pattern. The model considered here is described by the matrix $\hat{C}$ in which the pixel–pixel correlation is the same for all pairs of pixels and is equal to $\frac{\lambda}{N}$. The statistical properties of the model depend on three parameters: the reduced number of the stored patterns α, the temperature T and the reduced number of strength correlations of the pixels λ. The phase diagram in the space of parameters λ and α at temperature $T = 0$ is obtained. The network can retrieve patterns at $T = 0$ for $\alpha < \alpha_c$, where $\alpha_c \simeq 0.14$ as for the usual Hopfield neural network, but there is a new transition line above which a new local minimum of the free energy arises. This minimum corresponds to a ferromagnetic ordering of the neurons. There is another additional minimum (between the next two lines) that corresponds to mixed ordering. We also find the region where the ferromagnetic state becomes the ground state of the system.

1. Introduction

Most real life applications of neural networks deal with spatially organized data. For mainly technical reasons, the main results for neural networks (e.g. working as associative memory, fast parallel processing learning and relearning, robustness against degradation [1]) were obtained in the absence of any spatial structure of the inputs.

Among recent studies of neural networks there has been an increase of interest in the influence of the internal structure of spatially organized data on the capacity and the retrieval quality. Thus, the properties of the perceptron-like neural networks supplied with the spatial correlations of data have been investigated in [3], where, using the classic method introduced by Gardner in [2], it was shown that the storage properties of the neural network only depend on the eigenvalues of the correlation matrix C, whose elements, C_{ij}, give the correlation between the components i and j of the pattern ξ^{μ}. In particular, the critical capacity at zero stability always gives $\alpha_c = 2$.

A fully connected neural network which stores self-correlated patterns from an auto-associative standpoint has been considered in [3]. The results indicate that the storage capacity increases but the total quantity of information seems to decrease. The synaptic weights exhibit a structure richer than the one that they have for uncorrelated patterns. In addition to the usual $O(\frac{1}{\sqrt{N}})$ fluctuation weights, there is a O(1) self-averaging and ferromagnetic background. These short-range couplings take advantage of spatial

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correlations of the input patterns to enhance the local fields, without affecting the retrieval performances of large networks too much.

In this paper, we focus on the effects of such spatial organizations of the input patterns on the properties of Hopfield-like neural network [5, 7]. We consider the fully connected network consisting of N Ising spins $\{\sigma_i\}$, $(i = 1, \ldots, N)$ and symmetric couplings J_{ij} that are supposed to store p patterns $\{\xi_i^\mu\}$. The internal structure of data is given by the correlation matrix $\hat{C}$ whose elements C_{ij} are the same for all the pairs i, j and are equal to $\frac{\lambda}{N}$.

The model is studied in the limit when both $N \to \infty$ and $p \to \infty$, while the parameters $\alpha = \frac{p}{N}$ and λ remain finite. Such systems can be described by a Hamiltonian and could be studied in terms of the usual statistical mechanics. The specific form of the matrix $\hat{C}$ ($C_{ij} = \frac{\lambda}{N} \forall i, j$) does not correspond to any physical situation, but this model is convenient for studying the influence of the correlation on the retrieval quality of the neural networks because it is solvable and this solution is simple and not trivial.

On the other hand, a model with a more realistic correlation matrix C may be reduced to our model as a first approach of the mean-field approximation, like the mean-field approximation for usual ferromagnetic systems [6].

The main problem that we attempt to solve is the determination of the structure of the free-energy landscape, i.e. we are interested in the local minima and the ground states of the system.

The plan of the paper is the following. In section 2 we introduce the model and discuss the probability measure that should describe the model. In the section 3 we obtain the mean-field equations of the model. The phase diagram of the system in the space parameters λ and α, at $T = 0$, is obtained by a numerical solution of the saddle-point equations in section 4.

2. The model

The model consists of N Ising spins σ_i $(i = 1, \ldots, N)$ and is described by the Hamiltonian

$$H = -\tfrac{1}{2} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j \,. \tag{1}$$

The interaction matrix is taken to be of the form

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \tag{2}$$

where $\underline{\xi}^\mu$ $(\mu = 1, \ldots, p)$ are quenched patterns.

We choose a probability distribution for the patterns as follows:

$$\begin{aligned} &\langle\langle \xi_i^\mu \rangle\rangle = 0 \qquad \forall i \quad \forall \mu \\ &\langle\langle \xi_i^\mu \xi_j^\nu \rangle\rangle = \delta_{\mu\nu}(\delta_{ij} + C_{ij}) \qquad \forall (i, j) \quad \forall (\mu, \nu) \end{aligned} \tag{3}$$

where the angular brackets denote an average over this distribution.

The first equation (1) means that no external bias is taken, a situation which occurs, for example, when a pattern and its opposite are drawn with the same probability. The Kronecker symbol in (3) implies that different patterns are chosen independently of each other.

The matrix C contains the information about the correlations inside one pattern $\underline{\xi}^\mu$, and we choose it to obey the following requirements:

$$C_{ij} = \frac{\lambda}{N} \qquad \forall i, j \,. \tag{4}$$

We can construct a probability distribution satisfying the above conditions using the following argument. It is well known that any probability measure may be described by its characteristic function. A characteristic function and its probability measure are connected by the Fourier transformation

$$f(t_1,\ldots,t_N)=\int \mathrm{d}\xi_1\ldots\mathrm{d}\xi_N\exp\left(\mathrm{i}\sum_{k=1}^{N}t_k\xi_k\right)P(\xi_1,\ldots,\xi_N)$$

and the correlation function is given by

$$\langle\!\langle\xi_1^{k_1}\cdots\xi_N^{k_N}\rangle\!\rangle=\mathrm{i}^{\sum_{j=1}^{N}k_j}\left[\frac{\partial^{k_1+\cdots+k_N}f(t_1,\ldots,t_N)}{\partial t_1^{k_1}\cdots\partial t_N^{k_N}}\right]_{t_1=\cdots=t_N=0}.$$

In order to satisfy the conditions (3) we can choose the following characteristic function:

$$f(t_1,\ldots,t_N)=\left(\prod_{i=1}^{N}\cos(t_i)\right)\mathrm{e}^{-\frac{1}{2}\sum_{ij}(\delta_{ij}+C_{ij})t_it_j}.$$

3. The mean-field equation

The free energy of the model is calculated by using the replica method approach:

$$-\beta Nf=\lim_{n\to0}\frac{\langle\!\langle Z^n\rangle\!\rangle-1}{n} \tag{5}$$

where $\langle\!\langle\cdots\rangle\!\rangle$ means the averaging over the random ξ_i^μ and

$$Z^n=\sum_{\underline{\sigma}}\exp\left(\frac{\beta}{2N}\sum_{a=1}^{n}\sum_{\mu,i,j}\xi_i^\mu\xi_j^\mu\sigma_i^a\sigma_j^a\right) \tag{6}$$

is the replica partition function. Introducing the fields m_a^μ one gets

$$\langle\!\langle Z^n\rangle\!\rangle=\exp\left(-\frac{\beta pn}{2}\right)\left\langle\!\!\left\langle\int \mathrm{D}\underline{m}\sum_{\underline{\sigma}}\exp\left(-\tfrac{1}{2}\sum_{a,\mu}m_{a\mu}^2+\sqrt{\frac{\beta}{N}}\sum_{a,\mu,i}m_{a\mu}\sigma_i^a\xi_i^\mu\right)\right\rangle\!\!\right\rangle_{\xi_i^\mu} \tag{7}$$

where

$$\mathrm{D}\underline{m}=\left(\frac{1}{\sqrt{2\pi}}\right)^{np}\prod_{a\mu}\mathrm{d}m_{a\mu}. \tag{8}$$

Here we follow the standard calculations similar to those of the usual Hopfield model [7]. *We assume that only the overlap with pattern number one (i.e. $\underline{\xi}^1$) condenses in the $N\to\infty$ limit* and therefore we do the rescaling

$$m_{a1}\to\sqrt{\beta N}m_{a1}$$

$$\langle\!\langle Z^n\rangle\!\rangle=\exp\left(-\frac{\beta pn}{2}\right)(\beta N)^{\frac{n}{2}}\left\langle\!\!\left\langle\int \mathrm{d}\underline{m}_{a1}\,\mathrm{d}\underline{m}\sum_{\underline{\sigma}^a}\exp\left(-\tfrac{1}{2}\beta N\sum_a m_{a1}^2+\beta\sum_a m_{a1}\sum_i\sigma_i^a\xi_i^1\right.\right.\right.$$
$$\left.\left.\left.-\tfrac{1}{2}\sum_a\sum_{\mu=2}^{p}m_{a\mu}^2+\sqrt{\frac{\beta}{N}}\sum_{a,i}\sum_{\mu=2}^{p}m_{a\mu}\sigma_i^a\xi_i^\mu\right)\right\rangle\!\!\right\rangle_{\xi_i^\mu}. \tag{9}$$

After averaging over the random variables ξ_i^μ ($\mu \neq 1$) (see appendix A) we have

$$\langle\langle Z^n \rangle\rangle = \exp\left(-\frac{\beta p n}{2}\right)(\beta N)^{\frac{n}{2}} \left\langle\left\langle \int \mathrm{d}\underline{m}_{a1}\, \mathrm{d}\underline{m} \sum_{\underline{\sigma}} \exp\left(-\tfrac{1}{2}\beta N \sum_a m_{a1}^2 + \beta \sum_a m_{a1} \sum_i \sigma_i^a \xi_i^1 \right.\right.\right.$$
$$\left.\left.\left. -\tfrac{1}{2}\sum_a \sum_{\mu=2}^{p} m_{a\mu}^2 + \sum_{\mu=2}^{p} \frac{\beta}{2N} \sum_{a,b} m_{a\mu} m_{b\mu} \times \sum_{i,j} (\delta_{ij} + C_{ij}) \sigma_i^a \sigma_j^b \right) \right\rangle\right\rangle_{\xi_i^1} . \tag{10}$$

Let us consider the expression

$$\frac{1}{N} \sum_{i,j} (\delta_{ij} + C_{ij}) \sigma_i^a \sigma_j^b$$

from (4) we have

$$\frac{1}{N} \sum_{i,j} (\delta_{ij} + C_{ij}) \sigma_i^a \sigma_j^b = \delta_{ab} + (1 - \delta_{ab}) Q_{ab} + \lambda x_a x_b \tag{11}$$

where

$$Q_{ab} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^a \sigma_i^b \qquad \text{if} \quad a \neq b$$
$$Q_{ab} = 0 \qquad \text{otherwise} \tag{12}$$

and

$$x_a = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^a \tag{13}$$

$$\langle\langle Z^n \rangle\rangle = \exp\left(-\frac{\beta p n}{2}\right)(\beta N)^{\frac{n}{2}} \left\langle\left\langle \int \mathrm{d}\underline{m}_{a1}\, \mathrm{d}\underline{m}\, \mathrm{D}\underline{Q}\, \mathrm{D}\underline{x} \sum_{\underline{\sigma}} \exp\left(-\tfrac{1}{2}\beta N \sum_a m_{a1}^2 \right.\right.\right.$$
$$\left. + \beta \sum_a m_{a1} \sum_i \sigma_i^a \xi_i^1 - \tfrac{1}{2} \sum_{ab} \sum_{\mu=2}^{p} m_{a\mu} (\hat{I} - \beta \hat{I} - \beta \hat{Q} - \lambda \beta \hat{xx})_{ab} m_{b\mu} \right)$$
$$\left.\left. \times \prod_a \delta\left(N x_a - \sum_i \sigma_i^a \right) \prod_{a<b} \delta\left(N Q_{ab} - \sum_i \sigma_i^a \sigma_i^b \right) \right\rangle\right\rangle_{\xi^1} \tag{14}$$

where the matrix $\hat{Q}$ has been defined by (12) and the matrix $\hat{xx}$ has been defined as follows:

$$(\hat{xx})_{ab} = x_a x_b .$$

Introducing the conjugate fields R_{ab} and y_a for the Q_{ab} and the x_a, respectively, and integrating over $m_{a\mu}$ (with $\mu > 1$)

$$\langle\langle Z^n \rangle\rangle = \exp\left(-\frac{\beta p n}{2}\right)(\beta N)^{\frac{n}{2}} \left\langle\left\langle \sum_{\underline{\sigma}} \int \mathrm{d}\underline{m}_{a1} \mathrm{D}R\, \mathrm{D}\underline{Q}\, \mathrm{D}\underline{y}\, \mathrm{D}\underline{x} \right.\right.$$
$$\times \exp\left\{ -\tfrac{1}{2}\beta N \sum_a m_{a1}^2 + \beta \sum_a m_{a1} \sum_j \sigma_j^a \xi_j^1 \right.$$
$$- \frac{\alpha N}{2} \ln[\det((1-\beta)\hat{I} - \beta \hat{Q} - \lambda \beta \hat{xx})] + \mathrm{i}N \sum_{a<b} R_{ab} Q_{ab} \tag{15}$$
$$\left.\left.\left. + \mathrm{i}N \sum_a x_a y_a - \mathrm{i} \sum_{a<b} R_{ab} \sum_j \sigma_j^a \sigma_j^b - \mathrm{i} \sum_a y_a \sum_j \sigma_j^a \right\} \right\rangle\right\rangle_{\xi^1}$$

$$\langle\langle Z^n \rangle\rangle = \int \mathrm{D}m\, \mathrm{D}\underline{y}\, \mathrm{D}\underline{x}\, \mathrm{D}\underline{R}\, \mathrm{D}\underline{Q} \exp(-\beta N n f)$$

where f is the mean-field free energy of the model

$$f = \frac{1}{2n}\sum_a m_{a1}^2 + \frac{\alpha}{2n\beta}\ln[\det((1-\beta)\hat{I} - \beta\hat{Q} - \lambda\beta\hat{x}\hat{x})] + \frac{\alpha}{n}\sum_a x_a y_a + \frac{\alpha\beta}{n}\sum_{a<b} R_{ab}Q_{ab}$$
$$-\frac{1}{n\beta}\ln\left\langle\!\!\left\langle\left\{\sum_{\sigma^a}\exp\left(\beta\sum_a m_{a1}\sigma_j^a\xi_j^1 + \frac{\alpha\beta^2}{2}\sum_{a\neq b} R_{ab}\sigma_j^a\sigma_j^b + \alpha\beta\sum_a y_a\sigma_j^a\right)\right\}\right\rangle\!\!\right\rangle_{\xi^1}.$$

We have made the following rescalings:

$$R_{ab} \to \mathrm{i}\alpha\beta^2 R_{ab} \qquad y_a \to \mathrm{i}\alpha\beta y_a .$$

Here m_{a1} is the overlap with the condensed pattern

$$m_{a1} = \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i^a\rangle\xi_i^{\mu=1}$$

and Q_{ab} is the spin-glass parameter

$$Q_{ab} = \frac{1}{N}\sum_i\langle\sigma_i^a\sigma_i^b\rangle .$$

R_{ab} gives the average value of the noisy overlaps with non-condensed patterns

$$R_{ab} = \frac{1}{\alpha}\sum_{\mu=2}^{p} m_\mu^a m_\mu^b .$$

x_a is the parameter of 'ferromagnetic' ordering:

$$x_a = \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i^a\rangle .$$

Assuming replica symmetry one takes

$$R_{ab} = r \qquad a \neq b$$
$$Q_{ab} = q \qquad a \neq b$$
$$m_a = m \qquad y_a = y \qquad x_a = x .$$

Then taking the limit $n \to 0$ after some algebra one gets

$$f = \tfrac{1}{2}m^2 + \frac{\alpha}{2\beta}\left[\ln(1-\beta(1-q)) - \frac{\beta q + \lambda\beta x^2}{1-\beta(1-q)}\right] + \frac{\alpha}{2} + \tfrac{1}{2}\alpha\beta r(1-q) + \alpha x y$$
$$-\frac{1}{\beta}\langle\!\langle \ln\{2\cosh[\beta(m\xi + \alpha y + \sqrt{\alpha r}z)]\}\rangle\!\rangle_{z\xi^1} . \tag{16}$$

Here $\langle\!\langle\cdots\rangle\!\rangle$ means the averaging over ξ^1 and Gaussian z.
The corresponding saddle-point equations for the variables m, q, r, y and x are

$$m = \langle\!\langle \xi\tanh[\beta(m\xi + \alpha y + \sqrt{\alpha r}z)]\rangle\!\rangle_{z\xi^1} \tag{17}$$
$$q = \langle\!\langle \tanh^2[\beta(m\xi + \alpha y + \sqrt{\alpha r}z)]\rangle\!\rangle_{z\xi^1} \tag{18}$$
$$x = \langle\!\langle \tanh[\beta(m\xi + \alpha y + \sqrt{\alpha r}z)]\rangle\!\rangle_{z\xi^1} \tag{19}$$
$$y = \frac{\lambda x}{1-\beta(1-q)} \tag{20}$$
$$r = \frac{q + \lambda x^2}{[1-\beta(1-q)]^2} . \tag{21}$$

4. The phase diagram

At zero temperature (17)–(21) can be reduced to

$$m = \tfrac{1}{2}\mathrm{erf}\left(\frac{m+\alpha y}{\sqrt{2\alpha r}}\right) + \tfrac{1}{2}\mathrm{erf}\left(\frac{m-\alpha y}{\sqrt{2\alpha r}}\right) \tag{22}$$

$$x = \tfrac{1}{2}\mathrm{erf}\left(\frac{m+\alpha y}{\sqrt{2\alpha r}}\right) - \tfrac{1}{2}\mathrm{erf}\left(\frac{m-\alpha y}{\sqrt{2\alpha r}}\right) \tag{23}$$

$$c = \beta(1-q) = \sqrt{\frac{2}{\pi\alpha r}}\left(\tfrac{1}{2}\exp\left[-\frac{(m+\alpha y)^2}{2\alpha r}\right] + \tfrac{1}{2}\exp\left[-\frac{(m-\alpha y)^2}{2\alpha r}\right]\right) \tag{24}$$

$$y = \frac{\lambda x}{1-\beta(1-q)} = \frac{\lambda x}{1-c} \tag{25}$$

$$r = \frac{q+\lambda x^2}{[1-\beta(1-q)]^2} = \frac{1+\lambda x^2}{[1-c]^2} \tag{26}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}t\,\exp(-t^2)\,. \tag{27}$$

Introducing

$$u = \frac{m}{\sqrt{2\alpha r}} \qquad \text{and} \qquad v = \frac{y}{\lambda\sqrt{2\alpha r}}$$

and writing c, x and r as a function of u, v one gets

$$u = \frac{\phi_+(u,v)}{\left(\frac{2\alpha}{1-2\alpha\lambda v^2}\right)^{1/2} + \frac{2}{\sqrt{\pi}}Ex(u,v)} \tag{28}$$

$$v = \left(\frac{1}{2\alpha} - \lambda v^2\right)^{1/2} \phi_-(u,v) \tag{29}$$

where

$$\phi_+(u,v) = \tfrac{1}{2}\mathrm{erf}(u+\alpha\lambda v) + \tfrac{1}{2}\mathrm{erf}(u-\alpha\lambda v) \tag{30}$$

$$\phi_-(u,v) = \tfrac{1}{2}\mathrm{erf}(u+\alpha\lambda v) - \tfrac{1}{2}\mathrm{erf}(u-\alpha\lambda v) \tag{31}$$

$$Ex(u,v) = \tfrac{1}{2}\exp\left[-\tfrac{1}{2}(u+\alpha\lambda v)^2\right] + \tfrac{1}{2}\exp\left[-\tfrac{1}{2}(u-\alpha\lambda v)^2\right]. \tag{32}$$

These equations always have a solution $m = u = x = v = 0$ which corresponds to a spin-glass state with no macroscopic overlap with the learned patterns and non-zero spin-glass order parameter q given by (24)–(26) (with $m = x = y = 0$).

It is easy to see that if $v = 0$ equation (29) is automatically satisfied and (28) is reduced to an equation for the usual Hopfield model. Thus at $\alpha < \alpha_c = 0.138$ and any λ the solution with m different from 0 appears and it corresponds to the retrieval state. At $\alpha = \alpha_c$ there is a finite jump of the value of m from 0 to $m \neq 0$.

Instead, when $u = 0$ equation (28) is automatically satisfied and the simple analysis of (29) gives the line $\lambda_{fer}(\alpha) = \sqrt{\frac{\pi}{2\alpha}}$ above which the solution with $m = 0$ and $x \neq 0$ exists. This solution corresponds to the ferromagnetic state when the average site spin magnetizations $\langle\sigma_i\rangle$ have non-zero value x. The line $\lambda_{fer}(\alpha)$ is the line of the second phase transition.

The numerical solution of (28) and (29) shows that in the region restricted by the lines $\lambda_{m\downarrow}$ and $\lambda_{m\uparrow}$ the solution with $m \neq 0$ and $x \neq 0$ appears. This solution corresponds to

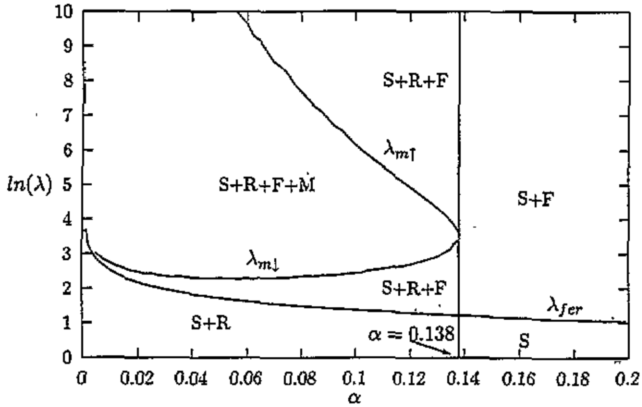

Figure 1. Zero-temperature phase diagram of the system. 'S'—spin-glass phase, 'R'—retrieval phase, 'F'—ferromagnetic' phase and 'M'—mixed phase.

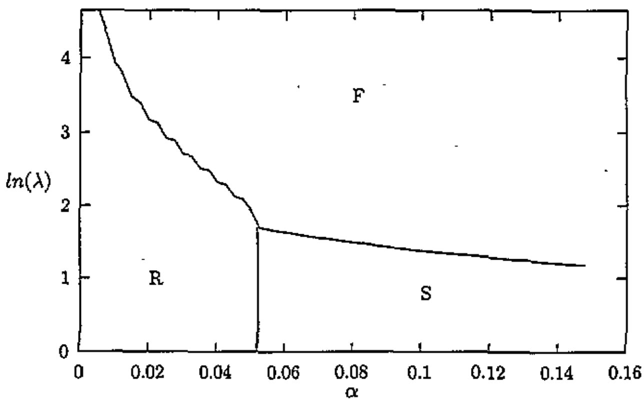

Figure 2. The structure of the ground states of the system.

the mixed state in which both the macroscopic overlap with the learned patterns and the parameter of ferromagnetic ordering are different from zero.

The phase diagram in the space of the parameters λ and α is shown in figure 1. In the region marked by 'S' the only stable state is the spin-glass state. In the region 'S + F' we have both the spin-glass state and the ferromagnetic state. In the region 'S + R' the spin-glass state and the retrieval state exist. The other denotations have analogous meanings, where 'M' denotes the mixed state. Thus in a region marked by 'S + R + F + M' all the phases are present.

The nature of the ground state of the system is shown in figure 2. In the region marked by 'S' the spin-glass state is the global minimum, in the region 'F' the ferromagnetic state is the global minimum. In the region marked by 'R' the retrieval state the global minimum.

5. Conclusion

We studied a simple Hopfield-like neural network model whose data have an internal structure that is described by weak long-range correlations between the components of the patterns. We observe that an effective ferromagnetic interaction arises in the system of spins due to this correlation. The effective interaction gives three new phase transition lines in the plane $0\alpha\lambda$, so the plane is divided into regions where beside the usual spin-glass and retrieval minima we have a new ferromagnetic and mixed minima.

It is well known that systems with weak long-range interactions do not have any regions with strong fluctuations and usually, for this reason, they are exactly solvable. The system considered above confirms these observations.

An interesting question arises from this point of view: if we study a neural network with short-range correlations (for example a neural network on a 2D square lattice where the correlation between two elements is different from 0 if the elements are the nearest neighbours) we certainly have to deal with a partition function on the 2D ferromagnetic Ising model, that presents a region of its parameters in which critical behaviour with strong fluctuations arises. The question mentioned above is: is this critical behaviour relevant for the properties of neural networks? Moreover, does this region of parameters of neural networks exists where strong fluctuations play the crucial role?

A detailed study of these questions will be reported elsewhere.

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Appendix

Here is the derivation for (9):

$$\langle\langle Z^n \rangle\rangle_{\xi_i^\mu} = \exp\left(-\frac{\beta p n}{2}\right)(\beta N)^{\frac{n}{2}} \left\langle\!\!\left\langle \sum_{\underline{\sigma}_i^a} \int \mathrm{d}m_a^1 \, \mathrm{d}\underline{m} \, \exp\left(-\tfrac{1}{2}\beta N \sum_a m_{a1}^2 \right.\right.\right.$$

$$\left.\left.\left. + \beta \sum_a m_a^1 \sum_i \sigma_i^a \xi_i^1 - \tfrac{1}{2} \sum_{a,\mu} m_{a\mu}^2 + \sqrt{\frac{\beta}{N}} \sum_{a,\mu,i} m_a^\mu \sigma_i^a \xi_i^\mu \right)\right\rangle\!\!\right\rangle_{\xi_i^\mu} .$$

The explicit computations are the following:

$$\left\langle\!\!\left\langle \exp\left(\sqrt{\frac{\beta}{N}} \sum_{a,\mu,i} m_a^\mu \sigma_i^a \xi_i^\mu\right)\right\rangle\!\!\right\rangle_{\xi^\mu} = \left\langle\!\!\left\langle \prod_{\mu=2}^{p} \exp\left(\sqrt{\frac{\beta}{N}} \sum_a m_a^\mu \sum_i \sigma_i^a \xi_i^\mu\right)\right\rangle\!\!\right\rangle_{\xi_i^\mu}$$

$$= \prod_{\mu=2}^{p} \left\langle\!\!\left\langle \exp\left(\sqrt{\frac{\beta}{N}} \sum_a m_a^\mu \sum_i \sigma_i^a \xi_i^\mu\right)\right\rangle\!\!\right\rangle_{\xi_i^\mu}$$

$$= \prod_{\mu=2}^{p} \left\langle\!\!\left\langle 1 + \sqrt{\frac{\beta}{N}} \sum_a m_a^\mu \sum_i \sigma_i^a \xi_i^\mu + \frac{\beta}{2N} \sum_{ab} m_a^\mu m_b^\mu \sum_{i,j} \sigma_i^a \sigma_j^b \xi_i^\mu \xi_j^\mu \right\rangle\!\!\right\rangle_{\xi_i^\mu}$$

$$= \prod_{\mu=2}^{p} \left(1 + \frac{\beta}{2N} \sum_{ab} m_a^\mu m_b^\mu \sum_{i,j} \sigma_i^a \sigma_j^b \langle\langle \xi_i^\mu \xi_j^\mu \rangle\rangle_{\xi_i^\mu}\right)$$

$$= \prod_{\mu=2}^{p} \left(1 + \frac{\beta}{2N} \sum_{ab} m_a^\mu m_b^\mu \left(\sum_i \sigma_i^a \sigma_i^b + \frac{\lambda}{N} \sum_{i \neq j} \sigma_i^a \sigma_j^b\right)\right)$$

$$= \prod_{\mu=2}^{p} \left(1 + \frac{\beta}{2N} \sum_{ab} m_a^\mu m_b^\mu \sum_{i,j} \sigma_i^a \sigma_j^b (\delta_{ij} + C_{ij})\right).$$

Then we have

$$\left\langle\!\!\left\langle \exp\left(\sqrt{\frac{\beta}{N}}\sum_{a,\mu,i} m_a^\mu \sigma_i^a \xi_i^\mu\right)\right\rangle\!\!\right\rangle_{\xi^\mu} = \left\langle\!\!\left\langle \exp\left\{\sum_\mu \frac{\beta}{2N}\sum_{a,b} m_{a\mu} m_{b\mu} \sum_{i,j}(\delta_{ij}+C_{ij})\sigma_i^a\sigma_j^b\right\}\right\rangle\!\!\right\rangle_{\xi^1} .$$

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